https://www.linkedin.com/feed/update/urn:li:activity:6539383812504334337

Prove that inequality

$$\log_e(e^\pi - 1) \cdot \log_e(e^\pi + 1) + \log_\pi(\pi^e - 1) \cdot \log_\pi(\pi^e + 1) < e^2 + \pi^2.$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

For any real $a, b > 1$ we have $\log_a(a^b - 1) \cdot \log_a(a^b + 1) =$

$\log_a\left(a^b \cdot \left(1 - \frac{1}{a^b}\right)\right) \cdot \log_a\left(a^b \cdot \left(1 + \frac{1}{a^b}\right)\right) =$

$\left(b + \log_a\left(1 - \frac{1}{a^b}\right)\right) \cdot \left(b + \log_a\left(1 + \frac{1}{a^b}\right)\right) =$

$b^2 + \log_a\left(1 - \frac{1}{a^b}\right) + \log_a\left(1 + \frac{1}{a^b}\right) + \log_a\left(1 - \frac{1}{a^b}\right) \cdot \log_a\left(1 + \frac{1}{a^b}\right) =$

$b^2 + \log_a\left(1 - \frac{1}{a^{2b}}\right) + \log_a\left(1 - \frac{1}{a^b}\right) \cdot \log_a\left(1 + \frac{1}{a^b}\right) < a^2$ (because

$\log_a\left(1 - \frac{1}{a^{2b}}\right) < 0$ and $\log_a\left(1 - \frac{1}{a^b}\right) \cdot \log_a\left(1 + \frac{1}{a^b}\right) < 0$).

By replacing (a, b) in inequality $\log_a(a^b - 1) \cdot \log_a(a^b + 1) < a^2$

with (b, a) we obtain $\log_b(b^a - 1) \cdot \log_b(b^a + 1) < b^2$ and, therefore,

$\log_a(a^b - 1) \cdot \log_a(a^b + 1) + \log_b(b^a - 1) \cdot \log_b(b^a + 1) < a^2 + b^2$.

In particular for $(a, b) = (e, \pi)$ we obtain inequality of the problem.